Appendix H
Formal Definitions

H.1. Planning problem

Formally, a planning problem is defined as a tuple $\Pi = \langle F, A, I, G \rangle$, where F is a finite set of propositions (or facts), A is a set of actions, $I \subseteq F$ is the *initial state*, and $G \subseteq F$ is the *goal*. Facts in a state are used to represent properties of the world. For example, a fact of the form $on(A, B)$ could be used to represent the fact that an object A is on top of an object B. A *state* s is a subset of F. The facts contained within a state are considered *true* and those not contained within a state are considered *false*. An *action* $a \in A$ is defined by 3 sets. $PRE(a)$, the *precondition*, which contains the facts that must hold true in order for action a to be executable; $ADD(a)$, facts that action a adds to the state when carried out; and $DEL(a)$, facts that action a deletes from the state when carried out (Kautz, McAllester & Selman, 1996). The resulting state from action a on s is defined as $\delta(s, a) = \big(s \setminus DEL(a)\big) \cup ADD(a)$. The state resulting from action a on s is defined as $\delta(s, a) = \big(s \setminus DEL(a)\big) \cup ADD(a)$. We extend the definition of δ for sequences of actions in the usual way. When a proposition p belongs to $ADD(a)$ we say that p is *added by* a. Furthermore, when p belongs to $DEL(a)$ we say that p is *deleted* by a.

An action a is defined as executable on a state s iff $PRE(a) \subseteq s$ (Muise et al., 2011). A sequence of actions $a\alpha$ is executable on a state s iff a is executable on s and α is executable on $\delta(s, a)$. Furthermore, we can define that if α is executable on s, then α achieves G from s iff α is executable on s and $G \subseteq \delta(s, \alpha)$. A sequential plan for Π is a sequence of actions α such that α achieves G from I. A *suffix* of a sequential plan $\alpha = [a_1, \ldots, a_n]$ is the sequence $[a_i, \ldots, a_n]$ where $1 \leq i \leq n + 1$. The *prefix* of the plan is defined analogously (Muise et al., 2011).

H.2. Sequential monitoring

Given a problem $\Pi = \langle F, A, I, G \rangle$ and a set of facts, ψ, expressed as a set of facts, the regression of ψ is defined in terms of an action a, denoted as $R(\psi, a)$, as follows: $R[\psi, a] = (\psi \setminus ADD(a)) \cup PRE(a),\ if\ ADD(a) \subseteq \psi$ y $DEL(a) \cap \psi = \emptyset$ (else $R[\psi, a]$ is undefined). The iterated regression on a sequence of actions α, denoted as $R^*[\psi, \alpha]$, is simply the successive application of the regression operator on each action within the sequence (assuming that this is defined for each step). For example, iff $\alpha = a_1, a_2, a_3$, then $R^*[\psi, \alpha] = R[R[R[\psi\ a_3], a_2], a_1]$ (Fritz & McIlraith, 2007). For the sake of readability, hereafter we shall use the notation R instead of R^*.

Based on the notion of regression proposed by Fritz et al., we define the *validity conditions* of a plan $\alpha = a_1 a_2 \ldots a_n$, denoted as $V_\Pi(\alpha)$, as the set $\{R^*[G, a_n], R^*[G, a_{n-1} a_n], \ldots, R^*[G, \alpha]\}$. We will therefore say that a plan α is *valid* on a state s iff there is a $\in V_\Pi(\alpha)$ such that $S \subseteq s$, i.e. a condition in $V_\Pi(\alpha)$ is met in s.

H.3. Partial order plans

A partial order O over a set of actions A defines a transitive, antisymmetric and reflexive relation $\preccurlyeq_O$, where we omit the subindex when it is clear from the context. A linearization of a set of actions A with regards to a partial order O over A is a sequence of actions $\alpha = a_1 a_2 \ldots a_n$ such that (1) each action of A appears only once in α and (2) if $a_i \preccurlyeq a_j \in O$ then $i \leq j$. We say that $a \prec b$ if and only if $a \preccurlyeq b$ and $a \neq b$. We denote the set of all the linearizations of A over O as $Lin(P)$. A tuple $P = \langle A, O\rangle$, where A is a set of actions and O is a partial order over A. P is a partial-order plan for Π if and only if every sequence in $Lin(P)$ is a plan for Π. Partial-order planning provides a compact representation of, in general, a number of linearizations which is exponential in the size of A.

H.4. Muise et al.'s (2011) method to partial-order plan

First, we must add the pair (G, P) to the list Γ in order to then start iterating on Γ. Three iterations will be performed as $|A| = 3$.

For iteration 1 we have that $\Gamma = \{(G, P)\}$ and $L = \emptyset$. $last(P)$ is computed to contain which actions could be executed last in P. Based on the graph of P (Figure 2), these are identified as the actions which no other actions depend upon, i.e. a_1 and a_2. $\Gamma = \{(R(G, a_1), P_1), (R(G, a_2), P_2)\}$, where P_1 and P_2 correspond to $prefix(P, a_1)$ and $prefix(P, a_2)$, respectively. In iteration 2, $\Gamma = \{(R(R(G, a_1), a_2), P'), (R(R(G, a_2), a_1), P')\}$ where $P' = prefix(prefix(P, a_1), a_2) = prefix(prefix(P, a_2), a_1) = \langle\{a_3\}, \emptyset\rangle$. The final iteration returns $\Gamma = \{(R(R(R(G, a_1), a_2), a_3), P''), (R(R(R(G, a_2), a_1), a_3), P'')\}$ where $P'' = \langle\emptyset, \emptyset\rangle$.

H.5. **Definition 1**

We say that $P_1 = \langle A_1, O_1\rangle$ and $P_2 = \langle A_2, O_2\rangle$ are independent with respect to the sets of propositions G_1 and G_2 if and only if:

1. No action in A_1 (respectively, A_2) has a delete that appears in G_2 (respectively, G_1) or in a precondition of an action in A_2 (respectively, A_1).
2. No fact added by an action in A_1 (respectively, A_2) is also deleted by an action in A_2 (respectively, A_1).
3. Furthermore, we say that $P_1, P_2, \ldots, P_n$ are independent if they are pairwise independent. This definition extends directly to the case in which either P_1 or P_2 are sequences of actions, in which a total order is assumed between actions.

H.6. **Definition 2**

Let $\alpha = a_1 a_2 \ldots a_n$ and $\beta = b_1 b_2 \ldots b_m$ be two sequences. The set of the two interleavings of α and β are inductively defined as $Int(\alpha, \beta) = \alpha$ if $m = 0$, $Int(\alpha, \beta) = \beta$ if $n = 0$, otherwise

$$Int(\alpha, \beta) = \{a_1\alpha' \mid \alpha' \in Int(a_2 \ldots a_n, \beta)\} \cup \{b_1\beta' \mid \beta' \in Int(\alpha, b_2 \ldots b_m)\}.$$

H.7. **Proposition 1**

Let $P_1 = \langle A_1, O_1\rangle$ and $P_2 = \langle A_2, O_2\rangle$. Then each linearization of α for $P_1 \cup P_2$ belongs to $Int(\alpha_1, \alpha_2)$, where α_1 and α_2 are linearizations of P_1 and P_2, respectively.

Proof: The proof is by contradiction. Let β be a linearization of $P_1 \cup P_2$ that does not achieve the goal. Now let β_1 and β_2 be sub-sequences of β that only contain actions stemming from $A_1$and A_2, respectively. Then one of β_1 or β_2 must violate the order at O_1 (or at O_2), which would mean that β is not a linearization of $P_1 \cup P_2$.

H.8. **Lemma 1**

Let P_1 and P_2 be two partial-order plans for $\Pi_1 = \langle F, A, I, G_1 \rangle$ and $\Pi_2 = \langle F, A, I, G_2 \rangle$, respectively. Furthermore, let P_1 and P_2 be independent of G_1 and G_2. Then $P_1 \cup P_2$ is a plan for $\Pi = \langle F, A, I, G_1 \cup G_2 \rangle$.

***Proof*:** Given that P_1 and P_2 are plans for Π_1 and Π_2, any linearization of both plans achieves G_1 and G_2 from I, respectively. Now let us take a linearization α of $P_1 \cup P_2$. Given Proposition 1, α must be an interleaving of the two linearizations of P_1 and P_2, let us call them α_1 and α_2. Given the definition of independence, α is executable as no precondition of an action α_1 can be invalidated by an action of α_2 and vice-versa. Finally, from the definition of independence we also get that α achieves $G_1 \cup G_2$ as no proposition added by an action of α_1 is deleted by an action of α_2 and vice-versa.

These results tell us that if we take two independent plans then the union between them is also a plan for the union of their objectives. Furthermore, any linearization of the resulting plan is an interleaving of the linearizations of the independent plans. This, therefore, provides an efficient way of calculating the validity conditions of an interleaving. More specifically, it tells us that when a sequence of actions comes from interleaving the sequences α_1 and α_2 then the regression of the interleaving can be calculated by only looking at α_1 and α_2. This finding is particularly interesting as when working with independent plans it means that we can focus on specific parts of the regression to calculate the regression for the whole plan.

H.9. **Lemma 2**

Let α_1 and α_2 be sequences of actions independent from $G_1$and G_2. Let $G = G_1 \cup G_2$ and, finally, let β be a sequence of actions in $Int(\alpha_1, \alpha_2)$. Then $R(G, p) = R(G_1, \alpha_1) \cup R(G_2, \alpha_2)$.

***Proof*:** By induction on the size of β. The base case can be verified directly because $|\beta| = |\alpha_1| = |\alpha_2| = 0$, and $G = G_1 \cup G_2$. By definition, in that case, the regression of a set of facts for the empty sequence is the same set. Now, let us suppose that the theorem holds for any β such that $|\beta| \leq k$, and show that this is also true for $a\beta$ where a is an action. We have two cases: that a is the first action of α_1, and that a is the first action of α_2. We will only look at the case in which $\alpha_1 = a\alpha_1'$; this, therefore, implies that $\beta \in Int(\alpha_1', \alpha_2)$. We, therefore, have that:

$$R(G, a\beta) = R(R(G, \beta), a)$$

Using the inductive hypothesis, we obtain that:

$$\begin{aligned} R(G, a\beta) &= R(R(G_1, \alpha_1') \cup R(G_2, \alpha_2), a) \\ &= \left(R(G_1, \alpha_1') \cup R(G_2, \alpha_2)\right) \setminus add(a) \cup prec(a) \end{aligned}$$

Given the assumption of independence between α_1 and α_2 from G_1 and G_2 we have that $R(G_2, \alpha_2)$ can only contain propositions in G_2 or propositions that are a precondition of an action in α_2. Therefore, it cannot contain any element in $add(a)$. We can therefore rewrite the above as:

$$R(G, a\beta) = (R(G_1, \alpha_1') \setminus add(a) \cup prec(a)) \cup R(G_2, \alpha_2)$$

Finally, using the definition of the operator R we get that $R(G, a\beta) = R(G_1, a\alpha_1') \cup R(G_2, \alpha_2)$. This concludes the proof of this lemma.

H.10. **Lemma 3**

Let $\alpha_1$and α_2 be sequences of actions independent of $G_1$and G_2. Furthermore, let α_1 and α_2 be plans for $\Pi_1 = \langle F, A, I, G_1\rangle$ and $\Pi_2 = \langle F, A, I, G_2\rangle$. Finally, let $\Pi = \langle F, A, I, G_1 \cup G_2\rangle$ and $\beta \in Int(\alpha_1, \alpha_2)$. Then $V_\Pi(\beta) = V_{\Pi_1}(\alpha_1) \otimes V_{\Pi_2}(\alpha_2)$.

***Proof*:**

$$\begin{aligned} V_\Pi(\beta) &= \{R(G, \beta') \mid \beta' \text{ is a suffix of } \beta\} \\ &= \{R(G, \beta') \mid \beta' \in Int(\alpha_1', \alpha_2'), \alpha_1' \text{ is a suffix of } \alpha_1, \alpha_2' \text{ is a suffix of } \alpha_2\} \end{aligned}$$

Using Lemma 2 we get that:

$$\begin{aligned} V_\Pi(\beta) &= \{R(G_1, \alpha_1') \cup R(G_2, \alpha_2') \mid \alpha_1' \text{ is a suffix of } \alpha_1, \alpha_2' \text{ is a suffix of } \alpha_2 \\ &= \{R(G_1, \alpha_1') \mid \alpha_1' \text{ is a suffix of } \alpha_1\} \otimes \{R(G_2, \alpha_2') \mid \alpha_2' \text{ is a suffix of } \alpha_2\} \\ &= V_{\Pi_1}(\alpha_1) \otimes V_{\Pi_2}(\alpha_2) \end{aligned}$$

H.11. **Theorem 1**

Let $\Pi = \langle F, A, I, G\rangle$ be a planning problem. Let $\{\Pi_i = \langle F, A, I, G_i\rangle\}_{i=1}^n$ be a sequence of planning problems such that $G = \bigcup_{i=1}^n G_i$, with $G_i \cap G_j = \emptyset$ when $i \neq j$. Let P_i be a plan for Π_i, for every $i \in \{1, \ldots, n\}$, such that $P_1, P_2, \ldots, P_n$ are independent of $G_1, G_2, \ldots, G_n$. Finally, let $P = \bigcup_{i=1}^n P_i = P$. Then $V_\Pi(P) = V_{\Pi_1}(P_1) \otimes V_{\Pi_2}(P_2) \otimes \cdots \otimes V_{\Pi_n}(P_n)$.

***Proof*:** Using the fact that $V_\Pi(P) = \bigcup_{\alpha \in Lin(P)} R(G, \alpha)$ we can see that all of $\alpha \in Lin(P)$ is an interleaving of linearizations of $P_1, P_2, \ldots, P_n$. We can then apply Lemma 3 to get the desired result.